Solutions

4.4: Game Theory
Solving Games, Reduction by Dominance, and Strictly Determined Games

---

Previously, we saw how to find an optimal counter-strategy when we already know the strategy of the other player. Next we will see how to find an "optimal strategy" with no knowledge of the other players moves. However, before proceeding to this point, consider the following.

Exercise 1. Let P be the payoff matrix for "rock, paper, scissors." That is to say

$$P = \begin{array}{c} \\ r \\ p \\ s \end{array}\begin{array}{c} \begin{array}{ccc} r & p & s \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}.$$

(a) Suppose player B plays scissors half the time and paper the other half. What is player A's optimal counter-strategy?

(b) Suppose player A uses the optimal counter-strategy from (a). What is player B's optimal counter-strategy?

(c) Suppose player B uses the optimal counter-strategy from (b). What is player A's optimal counter-strategy?

(d) If you were to repeat (b) and (c) repeatedly, always assuming that the other player is using the optimal counter-strategy from the previous stage, would the strategies tend towards a stable answer?

~~(e) From this exercise and your own experience playing "rock, paper, scissors," what do you think is the "optimal strategy?"~~

(a) $E = rPc = (x\ y\ z)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} = (-y+z)$ with $x+y+z=1$. Max at $z=1$, so $r^* = (0\ 0\ 1)$.

(b) $E = r^*Pc = (0\ 0\ 1)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-x+y)$ with $x+y+z=1$. Min at $x=1$, so $c^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(c) $E = rPc^* = (r\ p\ s)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (p-s)$ with $r+p+s=1$. Max at $p=1$, so $r^{**} = (0\ 1\ 0)$.

(d) The two strategies will just keep rotating through pure strategies rock → paper → scissors → rock
A plays scissors, so B plays rock, so A plays paper, so B plays scissors, etc...

Clearly this is not a good way to find the "optimal" strategy.

Minimax Criterion: The best strategy is the one that minimizes the maximum effect of a counterstrategy. In short, the best strategy minimizes risk.

Optimal Strategies

Exercise 1 helps to illustrate an important assumption which is common in many applications of game theory.

Fundamental Principle of Game Theory

Each player tries to use its best possible strategy, and assumes that the other player is doing the same.

Strategies found using this assumption will be referred to as optimal strategies.

Example 1. (Solving a 2×2 Game) Consider the payoff matrix

$$P = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}.$$

(a) Find the optimal strategy for the row player.

(b) Find the optimal strategy for the column player.

(c) Find the expected payoff of the game assuming both players use their optimal strategies.

(a) $E_1 = rPc_1 = [x \;\; 1-x]\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [-5x+3]$

$E_2 = rPc_2 = [x \;\; 1-x]\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [x-1]$

$E_1 = E_2$ when $x = \frac{2}{3}$. So optimal row strategy is $r^* = (\frac{2}{3}, \frac{1}{3})$.

(b) $E_1 = r_1 Pc = [1 \;\; 0]\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ 1-x \end{bmatrix} = [-2x]$

$E_2 = r_2 Pc = [0 \;\; 1]\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ 1-x \end{bmatrix} = [4x-1]$

$E_1 = E_2$ when $x = \frac{1}{6}$. So optimal column strategy is $c^* = \begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}$.

(c) $E^* = r^*Pc^* = [\frac{2}{3} \;\; \frac{1}{3}]\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix} = [-1/3]$. So A loses by 1/3 point or B wins by 1/3 point on average per game.

Finding the expected payoff, and hence the expected winner, of a game under the fundamental principle of game theory; i.e. when both players use their optimal strategies, is called solving the game. Thus, if you are ever asked to solve a 2×2 game, you are being asked to complete steps (a), (b), and (c) in example 1.

Reduction by Dominance

We have seen how to solve a 2×2 game. The following two strategies will allow us to solve some particular games of larger dimensions. Unfortunately, we will not see how to solve *every* game of larger dimensions. This further study into game theory is one possibility for study once we have met the course requirements.

In some instances, there are certain moves that are worse than other moves regardless of what strategy the opponent uses. We use a procedure known as reduction by dominance to remove these worse moves from play and reduce the payoff matrix to smaller dimensions.

Question 1. Let a_1 and a_2 be two different moves that the row player A can make. When is a_1 not viable (or wise) compared to a_2? Let b_1 and b_2 be two different moves that the column player B can make. When is b_1 not viable (or wise) compared to b_2?

a_1 is worse than a_2 if the entries are smaller pointwise. We say a_2 dominates a_1.

b_1 is worse than b_2 if the entries are larger pointwise. We say b_2 dominates b_1.

Example 2. Use reduction by dominance to reduce the payoff matrix for RTV and CTV given in example 3 of the previous handout.

	Nature Doc	~~Symphony~~	Ballet	~~Opera~~
~~Sitcom~~	~~2~~	~~1~~	~~-2~~	~~2~~
~~Docudrama~~	~~-1~~	~~1~~	~~-1~~	~~2~~
Reality Show	-2	~~0~~	0	~~1~~
Movie	3	~~1~~	-1	~~1~~

Ballets bring more viewers than Symphony or Opera regardless of what RTV plays.

Once Symphony and Opera are no longer options,

Movies bring more viewers than Sitcoms or Docudramas regardless of whether CTV plays Nature Docs or Operas.

Reduced Matrix = $\begin{array}{c c} & \text{Nature Doc} \quad \text{Opera} \\ \begin{array}{c}\text{Reality Show} \\ \text{Movie}\end{array} & \begin{pmatrix} -2 & 0 \\ 3 & -1 \end{pmatrix} \end{array}$ with no more dominant rows or columns.

After reduction by dominance, we can solve the 2×2 game that is remaining, as done in example 1, to solve this particular 4×4 game.

Example 1 solves this exact game. This means that assuming the Fundamental Principle of Game Theory (FPGT) CTV will gain 333 viewers

Strictly Determined Games

A game (of any size) is called strictly determined if the optimal strategies are both pure strategies. We will use a procedure of finding the row minima and column maxima to determine this. If there is an entry of the payoff matrix that is both the row minimum and the column maximum, then we refer to this entry as the saddle point. When there is a saddle point, it determines the optimal pure strategies.

Example 3. Solve the following game:

$$\begin{bmatrix} \boxed{-4} & -3 & 3 \\ 2 & -1 & \boxed{-2} \\ 1 & \boxed{0} & 2 \end{bmatrix}$$

Row Minima = $\square$

Column Maxima = $\diamond$

The (3,1)-entry is the saddle point.

It tells us that the expected payoff under FPGT is 0 with optimal pure strategies $r^* = (0\ 0\ 1)$ and $c^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Notice

$$E^* = r^* P c^* = (0\ 0\ 1)\begin{pmatrix} -4 & -3 & 3 \\ 2 & -1 & -2 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0).$$

A General Strategy for Solving Games

1. Reduce by dominance. This should always be your first step.
2. If you were able to reduce to a 1×1 game, you're done. The optimal strategies are the corresponding pure strategies, as they dominate all the others.
3. Look for a saddle point of the reduced game. If it has one, the game is strictly determined, and the corresponding pure strategies are optimal.
4. If your reduced game is 2×2 and has no saddle point, use the method of example 1 to find the optimal mixed strategies.
5. If your reduced game is larger than 2×2 and has no saddle point, your have to use linear programming to solve it, but this will have to wait until chapter 5.